ON THE CONSTRUCTION OF BODIES OF OPTIMUM SHAPE IN A SUPERSONIC STREAM

(K POSTROENIIU TEL OPTIMAL'NOI FORMY V SVERKHZVUKOVOM POTOKE)

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The determination of bodies of minimum drag and of nozzles with maximum flux for a given size was considered in the works [1] to [7]. Various schemes of solution were found, and the regions of their application in the hodograph plane. The determination of the regions of existence of these solutions in the flow plane requires the use of numerical methods, and has so far not been carried out. Insufficient attention to this side of the question leads to the loss of some solutions. The latter may contain portions of the boundary extremum prescribed by the limited dimensions of the bodies.

Below, the regions of existence of various solutions in the flow plane are determined and new schemes of solution are developed. The basic considerations are carried out on the example of nozzles.

1. Let it be required to construct the contour ab of the supersonic part of a plane or axisymmetric nozzle (Fig. 1a), possessing maximum thrust for a given position of the initial point a, length X, and maximum lateral dimension Y. Cartesian coordinates in the flow plane are denoted by x, y; in the axisymmetric case the x-axis coincides with the axis of symmetry. The contour of the entrance portion ma of the nozzle, completely determining the sonic line sq, is also given. A limitation may also be imposed upon the curvature of the initial portion of the contour ab.

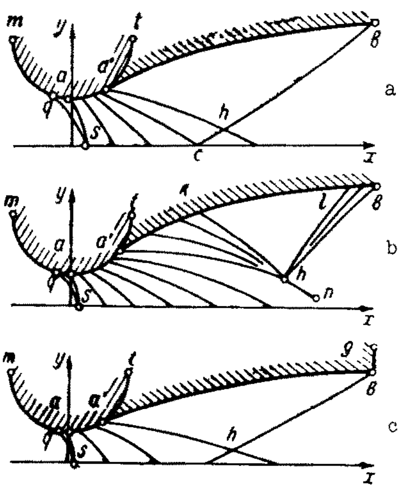

Fig. 1

In the solutions obtained up till now, the terminal portion of the contour a realizes a two-sided extremum.

Conversion to the characteristic control contour transforms the problem to the general problem of Lagrange with one independent variable. A continuous solution (in the sense of continuity of the function on the characteristic of cb) was obtained in the works [1] to [3], [6] and [7], and a solution with isentropic discontinuities in the works [5] to [7]. In the latter case (Fig. 1b) the contour ab contains two portions of the boundary extremum aa' and dk. The first of them is bounded by the prescribed curvature of the contour, and the second involves such a retardation of the stream that a shock wave hn begins on the boundary of the region of influence.

Aside from the portions just considered, the nozzle contour may contain portions of the boundary extrema $x = X$ and $y = Y$ associated with the limiting dimensions of the nozzle. We consider the portion $x = X$. With the presence of the step bg it is necessary in determining the force χ acting on the wall of the nozzle, to know the pressure distribution along this line (Fig. 1c). For flow of gas into a vacuum that problem is readily solved. For nonzero external pressure a stagnation zone forms at the step, and calculation of the flow becomes a very difficult problem. Here, consideration will be limited to the simplest case of constant pressure p_T on the portion bg, independent of the shape of the contour ag. Study of this scheme leads to the conclusion that introduction of a step under specific conditions permits the thrust of the nozzle to be increased.

To within a constant multiplier

$$\chi = \int_{y_a}^{y_g} p y^\nu dy$$

where p is the pressure. The quantities χ and X are expressed by integrals along the characteristics ac and cb and along the line bg (with the coinciding of a and a')

$$\chi = \int_{\psi_a}^{\psi_c} \left[w \cos\vartheta - \frac{p \sin(\vartheta-\alpha)}{\rho w \sin\alpha}\right] d\psi - \int_{\psi_b}^{\psi_c} \left[w \cos\vartheta + \frac{p \sin(\vartheta+\alpha)}{\rho w \sin\alpha}\right] d\psi + p_T \int_{y_b}^{y_g} y^\nu dy \qquad (1.1)$$

$$X = -\int_{\psi_a}^{\psi_c} \frac{\cos(\vartheta-\alpha)}{y^\nu \rho w \sin\alpha} d\psi - \int_{\psi_b}^{\psi_c} \frac{\cos(\vartheta+\alpha)}{y^\nu \rho w \sin\alpha} d\psi + \int_{y_b}^{y_g} x' dy \qquad (1.2)$$

where ψ is the stream function referred to $y_a^{\nu+1}\rho^* w^*$; ρ is the density referred to ρ^*, w is the velocity module referred to w^*, ϑ is the angle of inclination of the velocity vector to the stream axis, α is the Mach angle, ν is 0 or 1 for the plane and axisymmetric cases, respectively, the derivative $x' = dx/dy$ is taken along the contour of the body, and w^* and ρ^* are the critical values of velocity and density, respectively.

To solve the problem we use the method of Lagrange multipliers. A functional J, which includes the expressions for χ and X and differential relations on the closing characteristic bc is formed. From the expression for the first variation of J it follows that by virtue of the constancy of p_T the portion ab should be optimum also for a fixed position of the point b; that is, the shape of the contour ab is determined by the same equations as in the absence of the step bg. Variation of the coordinates of the point b with consideration of the necessary extremum conditions for the portion ab gives

$$\delta\chi = y_b^\nu (p - p_T - \rho w^2 \tan\alpha \sin\vartheta \cos\vartheta)_b \, \delta y_b + y_b^\nu (p_b - p_T) \tan\vartheta \delta x_b \qquad (1.3)$$

where δy_b and δx_b are the variations of the coordinates of point b.

If there is a solution with a portion bg, the value of δy_b is arbitrary, and $\delta x_b \leqslant 0$. From (1.3) follow the necessary conditions for an extremum of χ

$$p_b - p_T - \rho_b w_b^2 \tan\alpha_b \sin\vartheta_b \cos\vartheta_b = 0, \quad (p_b - p_T) \tan\vartheta_b \geqslant 0 \qquad (1.4)$$

The first equation determines the ordinate of point b. It is called the Busemann condition and was obtained previously [1], [6] and [7] in the problem of determining the optimum nozzle for a free transverse dimension. The second condition has, with consideration of (1.4), the form

$$\rho_b w_b^2 \operatorname{tg} \alpha_b \sin^2 \vartheta_b \geqslant 0 \tag{1.5}$$

In the absence of a step the permissible δy_b are negative. The necessary condition for a maximum of χ in this case is

$$p_b - p_T - \rho_b w_b^2 \operatorname{tg} \alpha_b \sin \vartheta_b \cos \vartheta_b \geqslant 0 \tag{1.6}$$

In the case of non-fulfillment of this condition, replacement of the contour by a nearby contour with a step lead to increase of the thrust χ. A cylindrical portion of the nozzle contour $y = Y$ is possible only when X exceeds the minimum nozzle length that gives a uniform exit stream for a given value of Y. However, in this case there exist an infinite number of solutions with the same value of thrust.

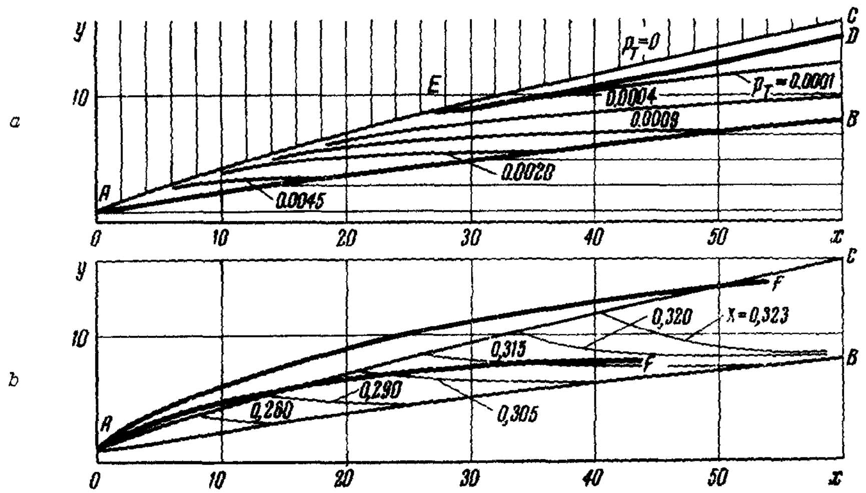

Fig. 2

2. According to what has been said, calculations for the optimum axisymmetric nozzle with a plane transition surface were accomplished. In the calculations a perfect gas has an adiabatic exponent $\varkappa = 1.4$. The departure from the transition surface was effected by expansion in series [9], and the calculation of the transonic flow by the method of characteristics [8]. The extremal characteristic was found from the relations of the work in [6].

The results of the calculations are shown in Fig. 2*a* and 2*b*. Fig. 2*a* shows the regions for various solutions in the flow plane. The line AB represents the geometric locus of the ends of nozzle contours of minimum length for uniform exit flow. The narrow lines represent the geometric loci of terminal points b at which the condition (1.4) is satisfied for various values of p_T. For example, line AC corresponds to $p_T = 0$. The points b belong to the region $DEAB$ for continuous solutions. The points b belong to the region DEC for solutions with isentropic discontinuities.

Various cases of the position of the terminal nozzle point are possible. If the given terminal nozzle point lies between the curve AB and the straight line $y = 1$, the optimum contour will consist of the contour giving a uniform stream and ending on AB, and of the cylingrical portion $y = Y$. If point b lies between the line for the given value of p_T and line AB, solution of the problem gives a contour reaching that point. Finally, if the terminal point lies below the corresponding line $p_T = \text{const}$, the nozzle must have a step portion bg. The point b lies on the intersection of the curve $p_T = \text{const}$ with the straight line $x = X$.

In Fig. 2*b* the narrow lines in the region CAB are the geometric loci of the ends of the optimum nozzles without steps for a constant value of χ. The heavy lines AF show two examples of nozzle contours.

3. We consider the $\alpha\vartheta$ plane. At the point b the quantities p, ρ and w are functions of α. In the plane $\alpha\vartheta$ conditions (1.4) and (1.6) determine the regions that must correspond to the terminal points of the extremal characteristics for different values of p_T.

In Fig. 3 the regions VMU corresponds to conditions (1.4) and (1.6) for $p_T = 0$ and region LNU for $p_T = 0.0778$. The value of $\varkappa = 1.4$. Let the terminal point of the desired contour be given. We try to find a solution in which the terminal point b of the extremal characteristic coincides with the given point. If the values α_b and ϑ_b obtained do not belong to the region (1.6), the contour that is found can be varied so that the thrust χ is increased. In this case the desired contour consists of the smooth portion ab and a step portion bg. The portion ab is determined by the necessary conditions for an extremum at $x_b = X$, where α_b and ϑ_b must satisfy condition (1.4).

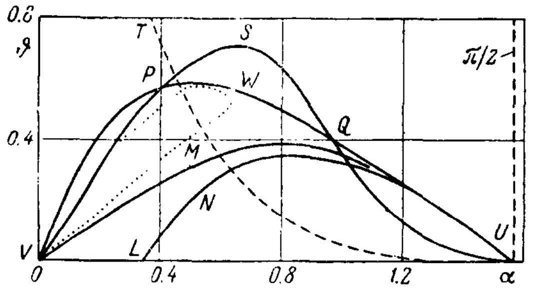

Fig. 3

Conditions (1.4) and (1.6) distinguish from all extremals those that give the solution of the problem. The minimum possible value of $p_T = 0$ and consequently the region of possible α_b and ϑ_b is bounded. In Fig. 3 below the curves VSU and VWU lies the region in which the extremal characteristics hb satisfy the necessary conditions for maximum thrust [6]. The dotted line bounds the region in which the points h of extremal characteristics hb lie after an isentropic discontinuity [5]. The dashed line UT gives the relation between α and ϑ at point a for Prandtl-Meyer flow. Calculations show [9] that this line is the upper limit of the regions corresponding to the fan of characteristics for flow past a corner a in the axisymmetric case.

This is true also in plane flow with a plane transition surface.

The quantities α and ϑ in plane isentropic flow are constant on the extremal characteristic. Therefore, from the relative location of the curves in Fig. 3, it follows that for $\varkappa = 1.4$ in plane nozzles a continuous solution is realized with a step portion or without one. In the axisymmetric case motion along the extremal from h to b corresponds to motion in the $\alpha\vartheta$ plane in the direction toward the axis $\vartheta = 0$. Therefore here are also realized solutions with isentropic discontinuities, to which the region CED corresponds (Fig. 2a). With increasing p_T the region of such solutions contracts and then disappears.

4. An analogous investigation can be carried out also in the case of external flow. In this case Fig. 3 is replaced by its mirror image in the α-axis, and the relations (1.4) to (1.6) are replaced by

$$p_b - p_T + \rho_b w_b^2 \operatorname{tg} \alpha_b \sin \vartheta_b \cos \vartheta_b = 0 \tag{4.1}$$

$$-(p_b - p_T) \tan \vartheta_b \geqslant 0 \tag{4.2}$$

$$p_b - p_T + \rho_b w_b^2 \operatorname{tg} \alpha_b \sin \vartheta_b \cos \vartheta_b \geqslant 0 \tag{4.3}$$

Equation (4.1) was obtained previously in the solution of the problems of a free transverse dimension [10] and [7]. From (4.3) it follows that the step is absent for $p_T < p_b$ and $\vartheta_b > 0$. In the case of isentropic axisymmetric flow the solutions with isentropic discontinuities are not realized, because moving along the extremal from h toward b in the $\alpha\vartheta$ plane leads to moving away from the axis $\vartheta = 0$.

For three plane profiles in the case of a uniform free stream parallel to the x-axis at Mach number $M = 2.858$ with $p_T = 0$ and $Y/X = -0.5530$, calculation of the drag coefficient c_x (C_D) gave the following results:

$$c_{x_1} = 0.1441, \qquad c_{x_2} = 0.1640, \qquad c_{x_3} = 0.1628$$

Profile 1 gives the solution of the problem. It consists of a rectilinear portion with $\vartheta = -0.1977$ and a step portion. Profile 2 is rectilinear and joins the points a and g. Profile 3 joins the points a and g and realizes the solution with an isentropic discontinuity. The value of c_x is determined by Equation

$$c_x = -2(\chi + p_\infty Y) / Y\rho_\infty w_\infty^2$$

where the subscript ∞ denotes parameters of the free stream.

5. We have here found the regions for different solutions in the flow plane, considered the hodograph plane, and found solutions with step portions under the assumption that the pressure of the step is constant and does not depend upon the shape of the desired contour. However, the pressure on the step ordinarily depends upon the shape of the contour ab. The qualitative investigation in [7], in which the presence of a step was postulated, shows that the solution including this dependence differs from the solution found above. At the same time the inequalities (1.6) and (4.3) are preserved. This makes it possible to find the region in which a contour without a step gives the solution. For verifying (1.6) and (4.3) it is sufficient to use an approximate value of p_T obtained, for example, by the method of [11].

It is essential that if the solution under the assumption of p_T as constant, leads to a contour with a step portion, when p_T is chosen to have its smallest possible value, then in reality this contour, not being optimum, ensures a higher thrust than the solutions without a step obtained previously.

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